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On Guided Waves Created by Line Defects

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Abstract The propagation of guided waves in photonic crystal fibers (PCFs) is studied. A photonic crystal fiber can be regarded as a perfect two-dimensional photonic crystal (PC) with a line defect along the axial direction. Under the assumption that the background spectrum has gaps, we give a simple condition on the parameters of the medium and of the line defect, which ensures the rise of eigenvalues in a specified subinterval of the given gap of the photonic crystal fiber. Using the modified Combes-Thomas estimates, we prove that the eigenfunctions corresponding to the eigenvalues decay exponentially away from the line defect.

Keywords Maxwell's equations · Photonic crystal fibers · Band gap · Guided waves · Line defects · Combes-Thomas estimates

1 Introduction

Photonic crystals (PCs) are periodically structured dielectric media, which are designed to favor band gaps, i.e., monochromatic electromagnetic waves of certain frequencies can not propagate through these structures. The fact that photonic crystals exhibit band gaps that bear a resemblance to semiconductors is, naturally, of tremendous interest in physics [17]. Since the first proposals of a photonic band gap effect by Yablonovitch [29] and John [16], lots of applications have been studied. Among these applications, photonic crystal fibers (PCFs) as fundamental transmission media to guide electromagnetic waves have been intensively studied. See, e.g., [2, 3, 6, 7, 20, 21]. Photonic crystal fibers consist of a periodic array of two different optical transparent materials running through the length of the fibers

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with a central *line defect* which serve as the core for light guiding. Physically, guided waves (or guided modes) can be created in these structures, i.e., electromagnetic waves of certain frequencies which propagate along the line defects of these structures are exponentially decaying in the transversal directions. (Note that, however, it is not proven in this paper that these modes are truly propagating rather than forming bound states. For a detailed discussion on this subject, we refer to [23].)

To the best of our knowledge, although this phenomenon has been intensively studied in physical experiments and numerical simulations, theoretical studies are few. Similar problems were studied in various circumstances. For instance in [9–11] they studied the localization phenomenon in 3D photonic crystals, in particular they proved the existence of bound states, confinements of impurity modes, and so on. As a closed related result, a line defect in the 2D photonic crystal was studied [23], in which the existence of the impurity spectrum in the gap (of the unperturbed operator) and, the exponential decay of the generalized eigenfunctions were proved. Note that a different physical background was considered there in comparison with ours, we refer to [23] for details. Recently, in [28], the transverse electric (TE) and transverse magnetic (TM) cases were studied. More precisely, in TM case, a guided wave only has a longitudinal electric field and a purely transverse magnetic field. Similarly, in TE case, a guided wave only has a longitudinal magnetic field and a purely transverse electric field. By dealing with the two 2D scalar differential equations, they proved the exponential decay of the guided waves in the cladding.

We have previously given rigorous proofs of the stability of essential spectrum, i.e., line defects do not change the essential spectrum of the background spectrum of the 2D operator generated on the cross-section [25] (see also Theorem 2.2 in this paper), which plays a key role for studying eigenvalues created by line defects. In this paper we continue our study for understanding this phenomenon. Under the assumption that the spectrum of the background medium has gaps, we give a sufficient condition to ensure the rise of eigenvalues in a specified subinterval of the given gap of the background spectrum. Physically, this means that if the background spectrum has band gaps, it is possible to guide electromagnetic waves with suitable cores. We also prove that the eigenfunctions corresponding to the eigenvalues decay exponentially away from the line defect. To do so, a Combes-Thomas estimate is needed. It is worth noting that some techniques used in this paper are inspired by the works of Figotin and Klein [9–11] and Kuchment and Ong [23].

This paper is outlined as follows. In Sect. 2, we show that this problem can be treated as an eigenvalue problem about a family of noncompact self-adjoint operators. In Sect. 3, we give a constructive method to prove the existence of eigenvalues created by line defects in the gap of the background spectrum. The Combes-Thomas estimates are formulated in Sect. 4. Finally, in Sect. 5 we prove that the eigenfunctions corresponding to the eigenvalues decay exponentially away from the line defect.

2 Mathematical Formulation

We will first give rigorous descriptions of some special photonic crystals and photonic crystal fibers. We adapt the following notations:

$$\vec{x} = (x^\top, x_3)^\top \in \mathbb{R}^3, \qquad x = (x_1, x_2)^\top \in \mathbb{R}^2.$$

We consider a lossless inhomogeneous dielectric medium occupying the whole space $\mathbb{R}^3$. The measurable functions $\epsilon_0(\vec{x})$ and $\mu_0(\vec{x})$, which describe the medium are called electric

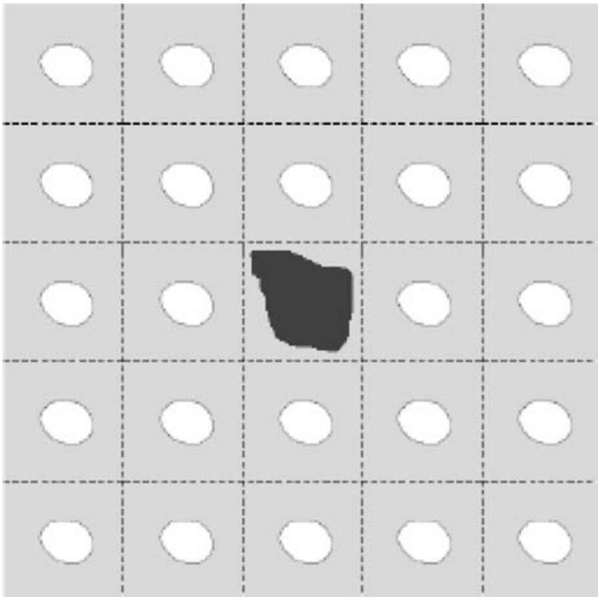

Fig. 1 The line defect is shown on the cross section of the photonic crystal fiber as a *darker region*

permittivity and magnetic permeability, respectively. We assume that $\epsilon_0(\vec{x})$ and $\mu_0(\vec{x})$ are invariant under any translation in the third normal direction x_3

$$\epsilon_0(\vec{x}) = \epsilon_0(x), \qquad \mu_0(\vec{x}) = \mu_0(x). \tag{1}$$

We also assume that there exist constants $\epsilon_{0,\pm}$ and $\mu_{0,\pm}$ such that

$$0 < \epsilon_{0,-} \le \epsilon_0(\vec{x}) \le \epsilon_{0,+} < \infty, \qquad 0 < \mu_{0,-} \le \mu_0(\vec{x}) \le \mu_{0,+} < \infty \quad \text{a.e.} \tag{2}$$

Such general conditions on $\epsilon_0(\vec{x})$ and $\mu_0(\vec{x})$, particularly the lack of smoothness, are required on physical grounds [10]. If they are periodic functions of the transverse variable x satisfying,

$$\epsilon_0(x+\vec{n}) = \epsilon_0(x), \quad \mu_0(x+\vec{n}) = \mu_0(x) \quad \text{for all } \vec{n} \in \mathbb{Z}^2,\ x \in \mathbb{R}^2, \tag{3}$$

these structures are often called (two-dimensional) photonic crystals, or photonic band gap materials [17] (see Fig. 1). *However, we don't require the functions* $\epsilon_0(x)$ *and* $\mu_0(x)$ *to satisfy the condition* (3) *in this paper unless stated otherwise*. Furthermore, a photonic crystal fiber is created if a line defect along x_3-direction is introduced. We describe the defect strip by

$$\tilde{\Omega}_l = \{\vec{x} = (x^\top, x_3)^\top \in \mathbb{R}^3 \mid x_3 \in \mathbb{R},\ x \in \Omega_l\} \quad \text{for } l > 0$$

where

$$\Omega_l \equiv l\Omega \tag{4}$$

is the support of the perturbation in the transverse plane. We assume that Ω is a measurable compact subset of $\mathbb{R}^2$. Without loss of generality, we also assume that the origin is an inner point of Ω. Inside the defect, the dielectric medium can be different from the background medium.

We use $\epsilon(\vec{x})$ and $\mu(\vec{x})$, independently of x_3-variable,

$$\epsilon(\vec{x}) = \epsilon(x), \qquad \mu(\vec{x}) = \mu(x), \tag{5}$$

to describe any media throughout this paper. Of course we do not require ϵ and μ to satisfy condition (3). The Maxwell's equations that govern the propagation of light and electromagnetic waves in the medium in absence of free charges and currents look as follows:

$$\begin{cases} \nabla_{\vec{x}} \times E(\vec{x},t) + \dfrac{\partial B(\vec{x},t)}{\partial t} = 0, & \nabla_{\vec{x}} \cdot B(\vec{x},t) = 0, \\ \nabla_{\vec{x}} \times H(\vec{x},t) - \dfrac{\partial D(\vec{x},t)}{\partial t} = 0, & \nabla_{\vec{x}} \cdot D(\vec{x},t) = 0, \end{cases} \tag{6}$$

where $E(\vec{x},t), H(\vec{x},t)$ are the electric and magnetic fields, and $D(\vec{x},t)$ and $B(\vec{x},t)$ are the displacement and magnetic induction fields, correspondingly. The so-called constitutive relations are

$$D(\vec{x},t) = \epsilon(x)E(\vec{x},t), \qquad B(\vec{x},t) = \mu(x)H(\vec{x},t).$$

We consider time-harmonic waves

$$E(\vec{x},t) = e^{i\omega t}\mathbb{E}(\vec{x}), \qquad H(\vec{x},t) = e^{i\omega t}\mathbb{H}(\vec{x}),$$

where $\omega > 0$ is the angular frequency. It leads from (6) to

$$\begin{cases} \nabla \times \mathbb{E}(\vec{x}) + i\omega\mu\mathbb{H}(\vec{x}) = 0, & \nabla \cdot (\mu\mathbb{H}) = 0, \\ \nabla \times \mathbb{H}(\vec{x}) - i\omega\epsilon\mathbb{E}(\vec{x}) = 0, & \nabla \cdot (\epsilon\mathbb{E}) = 0. \end{cases} \tag{7}$$

Since the functions $\epsilon(\vec{x})$ and $\mu(\vec{x})$ satisfy condition (5), guided waves are expected to propagate along x_3-direction in the medium. The rigorous definition of guided waves is as follows

Definition 2.1 *Guided waves* are solutions of (7) on the form

$$\begin{cases} \mathbb{E}(\vec{x}) = \big(E_1(x), E_2(x), E_3(x)\big)^\top e^{-i\beta x_3}, \\ \mathbb{H}(\vec{x}) = \big(H_1(x), H_2(x), H_3(x)\big)^\top e^{-i\beta x_3}, \end{cases} \tag{8}$$

and

$$\int_{\mathbb{R}^2} (\epsilon|E|^2 + \mu|H|^2)dx < \infty,$$

where

$$E = \big(E_1(x), E_2(x), E_3(x)\big)^\top, \qquad H = \big(H_1(x), H_2(x),\ H_3(x)\big)^\top$$

and $\beta > 0$ is the wave number in the x_3-direction.

We will introduce the following notation:

$$\nabla_\beta = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 0 \end{pmatrix} - i\beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ -i\beta \end{pmatrix},$$

where $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$. Furthermore, we define

$$\nabla_\beta \phi = (\partial_1 \phi, \partial_2 \phi, -i\beta\phi)^\top,$$

$$\nabla_\beta \times \vec{u} = (\partial_2 u_3 + i\beta u_2, -\partial_1 u_3 - i\beta u_1, \partial_1 u_2 - \partial_2 u_1)^\top \tag{9}$$

and

$$\nabla_\beta \cdot \vec{u} = \partial_1 u_1 + \partial_2 u_2 - i\beta u_3,$$

where $\vec{u} = (u_1, u_2, u_3)^\top$, and $\phi = \phi(x)$ is a scalar function.

Now plugging formula (8) into (7) and eliminating E or H, one obtains

$$\epsilon^{-1}\nabla_\beta \times \mu^{-1}\nabla_\beta \times E = \lambda E, \tag{10}$$

$$\mu^{-1}\nabla_\beta \times \epsilon^{-1}\nabla_\beta \times H = \lambda H, \tag{11}$$

where $\lambda = \omega^2$.

We first consider the E-formulation (10). In the following, some functional spaces are useful. We shall denote for any 3D vector field $\vec{u} = (u_1(x), u_2(x), u_3(x))^\top$ the transverse field by $u = (u_1(x), u_2(x))^\top$, thus we have $\vec{u} = (u^\top, u_3(x))^\top$. We define the scalar-valued operator

$$\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1,$$

and the space

$$H(\operatorname{curl}, \mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2; \mathbb{C}^2) \mid \operatorname{curl} u \in L^2(\mathbb{R}^2; \mathbb{C})\}$$

with the norm

$$\|u\|^2_{H(\operatorname{curl},\mathbb{R}^2)} = \|u\|^2_{L^2(\mathbb{R}^2)^2} + \|\operatorname{curl} u\|^2_{L^2(\mathbb{R}^2)}.$$

A standard Sobolev space is also needed

$$H^1(\mathbb{R}^2) = \{\phi \in L^2(\mathbb{R}^2; \mathbb{C}) \mid \nabla\phi \in L^2(\mathbb{R}^2; \mathbb{C}^2)\}.$$

Furthermore, we also define

$$H_\epsilon = L^2(\mathbb{R}^2; \mathbb{C}^3) \tag{12}$$

equipped with the weighted inner product

$$\langle \vec{u}, \vec{v}\rangle_\epsilon = \int_{\mathbb{R}^2} \epsilon \vec{u} \cdot \overline{\vec{v}} dx$$

and the norm $\|\vec{u}\|_\epsilon = \sqrt{\langle \vec{u}, \vec{u}\rangle_\epsilon}$, where $\overline{\vec{v}}$ means the conjugate of $\vec{v}$.

We introduce

$$V_\epsilon = \{\vec{u} \in H_\epsilon \mid \nabla_\beta \times \vec{u} \in H_\epsilon\}. \tag{13}$$

The space V_ϵ is a Hilbert space equipped with the norm

$$\|\vec{u}\|^2_{V_\epsilon} = \int_{\mathbb{R}^2} \epsilon(|\vec{u}|^2 + |\nabla_\beta \times \vec{u}|^2) dx.$$

In the following of this section we review some results obtained in [25]. We do not attempt to give the detailed proofs of them in this paper.

Lemma 2.1 *V_ϵ is isomorphic into $H(\mathrm{curl}, \mathbb{R}^2) \times H^1(\mathbb{R}^2)$ and the norm $\|\cdot\|_{V_\epsilon}$ is equivalent to the norm $\|\cdot\|_{H(\mathrm{curl},\mathbb{R}^2)\times H^1(\mathbb{R}^2)}$, i.e.,*

$$V_\epsilon = \{\vec{u} \mid \vec{u} = (u^\top, u_3)^\top \in H(\mathrm{curl}, \mathbb{R}^2) \times H^1(\mathbb{R}^2)\}.$$

As in [4, 18], we will give a formulation of the problem in which the divergence-free condition

$$\nabla_\beta \cdot (\epsilon E) = 0 \quad \text{for } E = (E_1, E_2, E_3)^\top \text{ satisfing (10) and } \lambda \neq 0$$

is included in the functional framework. With such a formulation we can work equivalently with the E-formulation (10) or the H-formulation (11) which allows us to take profit from the natural symmetry of Maxwell's equations with respective to E and H. Towards this goal, we shall give a modified Weyl's decomposition.

Lemma 2.2 *The space H_ϵ can be decomposed to the direct sum of the spaces $H_\epsilon(\beta)$ and $G(\beta)$*

$$H_\epsilon = H_\epsilon(\beta) \oplus G(\beta), \tag{14}$$

where

$$H_\epsilon(\beta) = \{\vec{u} \in H_\epsilon \mid \nabla_\beta \cdot (\epsilon \vec{u}) = 0\}$$

and

$$G(\beta) = \{\nabla_\beta \phi \mid \phi \in H^1(\mathbb{R}^2)\}.$$

The sum (14) is orthogonal with respect to the scalar product with the weight $\epsilon(x)dx$.

We define the Maxwell operator $\mathcal{A}_\epsilon(\beta) = \epsilon^{-1}\nabla_\beta \times \mu^{-1}\nabla_\beta \times$ as the nonnegative self-adjoint operator on $L^2(\mathbb{R}^2; \mathbb{C}^3)$, uniquely defined by the nonnegative quadratic form given as the closure of

$$\mathfrak{a}_\epsilon(\beta; \vec{u}, \vec{v}) = \int_{\mathbb{R}^2} (\mu^{-1}\nabla_\beta \times \vec{u}) \cdot \overline{(\nabla_\beta \times \vec{v})} dx, \quad \vec{u}, \vec{v} \in C_0^1(\mathbb{R}^2; \mathbb{C}^3).$$

We can describe the structure of $\mathcal{A}_\epsilon(\beta)$ by

Lemma 2.3

i) $\mathcal{K}\mathrm{er}\mathcal{A}_\epsilon(\beta) = G(\beta)$,
ii) $\mathcal{I}\mathrm{m}\mathcal{A}_\epsilon(\beta) \subset H_\epsilon(\beta)$.

Since $\mathcal{A}_\epsilon(\beta) \mid_{G(\beta)} = 0$, we have $\sigma(\mathcal{A}_\epsilon(\beta)) = \{0\} \cup \sigma(\mathcal{A}_\epsilon(\beta) \mid_{H_\epsilon(\beta) \cap V_\epsilon})$. It is natural to work on the restriction of $\mathcal{A}_\epsilon(\beta)$ to the space $H_\epsilon(\beta) \cap V_\epsilon$, i.e.,

$$A_\epsilon(\beta) := \mathcal{A}_\epsilon(\beta) \mid_{V_\epsilon(\beta)},$$

where

$$V_\epsilon(\beta) := H_\epsilon(\beta) \cap V_\epsilon = \{\vec{u} \in V_\epsilon \mid \nabla_\beta \cdot (\epsilon \vec{u}) = 0\} \tag{15}$$

with V_ϵ defined in (13). The closed quadratic form $a_\epsilon(\beta;\cdot,\cdot)$ corresponding to $A_\epsilon(\beta)$ is

$$a_\epsilon(\beta\,;\vec{u},\vec{v})=\int_{\mathbb{R}^2}(\mu^{-1}\nabla_\beta\times\vec{u})\cdot\overline{(\nabla_\beta\times\vec{v})}dx \quad \text{for all } (\vec{u},\vec{v})\in V_\epsilon(\beta)\times V_\epsilon(\beta). \tag{16}$$

Next, a two dimensional scalar-valued operator div is defined by

$$\operatorname{div} u=\partial_1 u_1+\partial_2 u_2 \quad \text{for } u=(u_1,u_2)^\top.$$

Theorem 2.1 *For any $\beta>0$, the operator $A_\epsilon(\beta)$ is self-adjoint, positive and*

$$\sigma(A_\epsilon(\beta))\subset[\rho_-\beta^2,\infty),$$

where

$$\rho_-=\inf_{x\in\mathbb{R}^2}\big(\epsilon^{-1}(x)\mu^{-1}(x)\big)>0. \tag{17}$$

Remark 2.1 Theorem 2.1 is just the first step for studying the spectral properties of $A_\epsilon(\beta)$. It is well-known that the spectrum of $A_\epsilon(\beta)$ consists of an essential spectrum corresponding to a continuum of radiating modes and a point spectrum corresponding to guided modes. Of course the radiating modes have no finite energy in the transverse plane.

In the sequel, we describe the background medium by ϵ_0 and μ_0, and the perturbed medium by $\tilde{\epsilon}(x)$ and $\tilde{\mu}(x)$. We adapt $A_{\tilde{\epsilon}}(\beta)$ and $\mathcal{A}_{\tilde{\epsilon}}(\beta)$ as the perturbed operator according to $A_{\epsilon_0}(\beta)$ and $\mathcal{A}_{\epsilon_0}(\beta)$, respectively. We also introduce

$$\eta(x)=\tilde{\mu}^{-1}(x)-\mu_0^{-1}(x), \qquad \xi(x)=\tilde{\epsilon}^{-1}(x)-\epsilon_0^{-1}(x) \tag{18}$$

and

$$\eta_\pm=\max\{\pm\eta(x),0\}, \qquad \xi_\pm=\max\{\pm\xi(x),0\}.$$

By our hypotheses (4), both ξ and η are supported inside Ω_l.

Under the assumption that both $\xi(x)$ and $\eta(x)$ do not change their signs a.e. $x\in\mathbb{R}^2$, we have

Theorem 2.2 (Stability of essential spectrum)

$$\sigma_{ess}(A_{\tilde{\epsilon}}(\beta))=\sigma_{ess}(A_{\epsilon_0}(\beta)).$$

This result means that the insertion of a line defect does not change the essential spectrum of the nonnegative operator $A_0(\beta)$ according to the background medium. It is a consequence of Weyl's theorem on the stability of the essential spectrum (see Sect. XIII.4 in [26]).

3 Existence of Eigenvalues Created by Line Defects

In this section we consider the case that the perturbed medium is homogeneous inside the line defect $\tilde{\Omega}_l$, more precisely,

$$\tilde{\epsilon}(x)=\begin{cases}\epsilon_1, & \text{if } x\in\Omega_l\\ \epsilon_0(x), & \text{if } x\notin\Omega_l\end{cases}, \qquad \tilde{\mu}(x)=\begin{cases}\mu_1, & \text{if } x\in\Omega_l\\ \mu_0(x), & \text{if },x\notin\Omega_l\end{cases},$$

where ϵ_1 and μ_1 are two positive constants which are assumed to satisfy

$$\epsilon_1 \geq \epsilon_0(x), \quad \mu_1 \geq \mu_0(x) \quad \text{a.e. } x \in \mathbb{R}^2, \tag{19}$$

and Ω_l is defined in (4). As we mentioned in Sect. 2, the background medium described by $\epsilon_0(x)$ and $\mu_0(x)$ which satisfy condition (3) is called a photonic crystal. Floquet-Bloch theory [22, 26] shows that the spectrum of the periodic operator $A_{\epsilon_0}(\beta)$ is the union of a countable number of bands; more precisely, there exist continuous periodic functions $\lambda_j(k)$, $j = 1, 2, 3, \dots$ on $\mathbb{R}^2$ with period 2π, such that

$$\sigma\big(A_{\epsilon_0}(\beta)\big) = \bigcup_{j=1}^{\infty} \lambda_j(Q)$$

where $Q = [-\pi, \pi)^2$ is called Brillioun zone in physical literature [17]. Bands can overlap and fill all the real semiaxis, or can be separated by the gaps. The existence of gaps in the spectra of the periodic Maxwell operators was studied by many authors, we refer to [12, 15] for scalar model, [13] for two-dimensional photonic crystals, and [8] for the full vectorial Maxwell operators. As a somewhat straightforward application of the high contrast results on existence of gaps developed in, e.g., [12, 13, 15], it was shown in [28] that an appropriate choice of β leads to an effective high contrast, thus gap can be created. (However, no attempt is made to give a complete survey on this subject here.)

Although it is usually assumed in the photonic crystal theory that $\epsilon_0(x)$ and $\mu_0(x)$ are both periodic functions, this is unnecessary for the basic results we get in this paper. In the sequel, we suppose that $B = (a, b)$ is a band gap in the spectrum of the operator $A_{\epsilon_0}(\beta)$ associated with the periodic background medium, where $0 < a < b < \infty$. For any $\tau \in B$, we can choose $d > 0$ such that

$$I_{\tau,d} \equiv [\tau - d, \tau + d] \subset B. \tag{20}$$

The following theorem is a development of Theorem 2 in [11] and Theorem 1 in [23].

Theorem 3.1 *Let $B = (a, b)$ be a band gap in the spectrum of the operator $A_{\epsilon_0}(\beta)$. For any interval $I_{\tau,\, d}$ satisfying (20), if at least one of the conditions below is satisfied,*

i) *for $\epsilon_1\mu_1$ fixed, and*

$$l^2 \geq \frac{2(\tau\epsilon_1\mu_1 - \beta^2)}{d^2\epsilon_1^2\mu_1^2} \inf_{\substack{\phi\in C_0^2(\Omega)\\ \|\phi\|=1}} \left(\|\vec{n}_k \cdot \nabla\phi\|^2 \left(1 + \sqrt{1 + \frac{d^2\epsilon_1^2\mu_1^2\|\Delta\phi\|^2}{4(\tau\epsilon_1\mu_1 - \beta^2)^2\|\vec{n}_k \cdot \nabla\phi\|^4}}\right)\right), \tag{21}$$

or

ii) *for l fixed, and*

$$\epsilon_1\mu_1 \geq \frac{2\tau}{l^2 d^2} \inf_{\substack{\phi\in C_0^2(\Omega)\\ \|\phi\|=1}} \Bigg(\|\vec{n}_k \cdot \nabla\phi\|^2 \times \left(1 + \sqrt{\max\left\{0, 1 + \frac{d^2(\|\Delta\phi\|^2 - 4l^2\beta^2\|\vec{n}_k \cdot \nabla\phi\|^2)}{4\tau^2\|\vec{n}_k \cdot \nabla\phi\|^4}\right\}}\right)\Bigg), \tag{22}$$

where $\vec{n}_k$ is a unit vector which parallels with the vector $\nabla\phi$, then the interval $I_{\tau,\, d}$ contains at least one eigenvalue of the perturbed operator $A_{\tilde{\epsilon}}(\beta)$.

We shall first give some comments on (21) and (22) before the proof. In fact we can give simple forms of conditions (21) and (22). Using the inequality

$$\sqrt{1+a} < 1 + \frac{a}{2} \quad \text{for } a > 0$$

and neglecting the term $-4l^2\beta^2\|\vec{n}_k \cdot \nabla\phi\|^2$ in (22), the conditions (21) and (22) can be simplified as

$$l^2 \geq \frac{\tau\epsilon_1\mu_1 - \beta^2}{d^2\epsilon_1^2\mu_1^2} \inf_{\substack{\phi\in C_0^2(\Omega)\\ \|\phi\|=1}} \left(4\|\vec{n}_k\cdot\nabla\phi\|^2 + \frac{d^2\epsilon_1^2\mu_1^2\|\Delta\phi\|^2}{4(\tau\epsilon_1\mu_1-\beta^2)^2\|\vec{n}_k\cdot\nabla\phi\|^2}\right)$$

and

$$l^2\epsilon_1\mu_1 \geq \frac{\tau}{d^2} \inf_{\substack{\phi\in C_0^2(\Omega)\\ \|\phi\|=1}} \left(4\|\vec{n}_k\cdot\nabla\phi\|^2 + \frac{d^2\|\Delta\phi\|^2}{4\tau^2\|\vec{n}_k\cdot\nabla\phi\|^2}\right),$$

respectively. For $\Omega = \{x\in\mathbb{R}^2 \mid |x|\leq 1\}$, we can further estimate by using $\phi = \sqrt{\frac{3}{\pi}}(1-(x_1^2+x_2^2))$ as an approximate function. Set

$$\vec{n}_k = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1, x_2)^\top,$$

one can easily verify that such a choice satisfies (25), (28) and (31). Simple calculations show that $\|\phi\| = 1$, $\|\vec{n}_k\cdot\nabla\phi\| = \sqrt{6}$ and $\|\Delta\phi\| = 4\sqrt{3}$. Thus (21) and (22) can be expressed on simple forms as

$$l^2 \geq \frac{24(\tau\epsilon_1\mu_1-\beta^2)}{d^2\epsilon_1^2\mu_1^2} + \frac{2}{\tau\epsilon_1\mu_1-\beta^2}$$

and

$$l^2\epsilon_1\mu_1 \geq \frac{24\tau}{d^2} + \frac{2}{\tau},$$

respectively.

Now we shall prove this theorem.

Proof By Theorem 2.2, we know that if $A_{\tilde{\epsilon}}(\beta)$ has spectrum inside the gap in the spectrum of $A_{\epsilon_0}(\beta)$, this spectrum must consist of isolated eigenvalues with finite multiplicity only. Since $\sigma(\mathcal{A}_{\tilde{\epsilon}}(\beta)) = \{0\}\cup\sigma(A_{\tilde{\epsilon}}(\beta))$, it is equivalent to prove the existence of eigenvalues of the operator $\mathcal{A}_{\tilde{\epsilon}}(\beta)$ in the interval $I_{\tau,d}$. Hence it suffices to prove this theorem if we can find an approximate eigenfunction $\vec{u}$ of the operator $\mathcal{A}_{\tilde{\epsilon}}(\beta)$ such that

$$\|(\mathcal{A}_{\tilde{\epsilon}}(\beta)-\tau)\vec{u}\|_{\tilde{\epsilon}} \leq d\|\vec{u}\|_{\tilde{\epsilon}}. \tag{23}$$

Since the medium is homogeneous inside the defect strip $\tilde{\Omega}_l$, we have

$$\begin{aligned}\mathcal{A}_{\tilde{\epsilon}}(\beta) &= (\epsilon_1\mu_1)^{-1}\nabla_\beta\times\nabla_\beta\times \\ &= (\epsilon_1\mu_1)^{-1}\nabla_\beta(\nabla_\beta\cdot) - (\epsilon_1\mu_1)^{-1}\nabla_\beta\cdot\nabla_\beta\otimes I_3 \\ &= (\epsilon_1\mu_1)^{-1}\nabla_\beta(\nabla_\beta\cdot) - (\epsilon_1\mu_1)^{-1}\Delta_\beta\otimes I_3 \end{aligned} \tag{24}$$

inside the defect strip, where I_3 is the identity operator on $\mathbb{C}^3$ and $\Delta_\beta = \Delta - \beta^2$ with $\Delta = \partial_1^2 + \partial_2^2$.

We set a real-valued cut-off function $\phi(x)$ such that $\phi(x) \in C_0^2(\Omega)$ and $\|\phi\| = 1$. Further, we introduce $\phi_l(x) \equiv \frac{1}{l}\phi(\frac{x}{l})$ for $l > 0$. One can easily see that $\|\phi_l\| = 1$. Then we will construct an approximate eigenfunction $\vec{u}_l$ in the following way:

$$\vec{u}_l := e^{ik_\beta \cdot \vec{x}} \phi_l(x)\zeta \quad \text{for } \vec{x} \in \mathbb{R}^3,$$

where $\zeta = (\zeta_1, \zeta_2, 0)^\top \in \mathbb{R}^3$, $|\zeta| = 1$ and ζ is chosen in such a way:

$$\zeta \cdot \nabla_\beta \phi = 0, \tag{25}$$

$k_\beta \equiv (k^\top, -\beta)^\top$ for $k = (k_1, k_2)^\top \in \mathbb{R}^2$ and

$$|k|^2 = \tau \epsilon_1 \mu_1 - \beta^2. \tag{26}$$

Since $\tau \in B$, in view of Theorem 2.1, we have $\tau \geq \rho_- \beta^2$, where ρ_- is defined in (17). Using the assumption (19), we have

$$\beta^2 \leq \tau \rho_-^{-1} \leq \tau \epsilon_1 \mu_1. \tag{27}$$

Hence (26) makes sense. We further choose k_β such that

$$k_\beta \cdot \zeta = 0. \tag{28}$$

In fact we can see from (25) and (28) that the vector k is parallel to the vector $\nabla\phi$. Obviously we can see $\vec{u}_l$ defined above belongs to the space $\mathrm{Dom}(\mathcal{A}_{\tilde{\epsilon}}(\beta))$ and also $\|\vec{u}_l\| = 1$. Since

$$\nabla_\beta \cdot (\vec{u}_l) = (\nabla_\beta \cdot \zeta)(\phi_l e^{ik_\beta \cdot \vec{x}}) = \zeta \cdot (\nabla_\beta \phi_l) e^{ik_\beta \cdot \vec{x}} + ik_\beta \cdot \zeta \phi_l e^{ik_\beta \cdot \vec{x}} = 0,$$

by applying (24) and (26) we have

$$\begin{aligned}
&\nabla_\beta \times \nabla_\beta \times \vec{u}_l \\
&\quad = -\Delta_\beta \otimes I_3 \vec{u}_l \\
&\quad = (-\Delta + \beta^2) \otimes I_3 \big(e^{ik_\beta \cdot \vec{x}} \phi_l(x)\zeta\big) \\
&\quad = \big(-\Delta(e^{ik_\beta \cdot \vec{x}})\phi_l - e^{ik_\beta \cdot \vec{x}} \Delta\phi_l - 2\nabla(e^{ik_\beta \cdot \vec{x}}) \cdot \nabla\phi_l + \beta^2 e^{ik_\beta \cdot \vec{x}} \phi_l\big)\zeta \\
&\quad = (\tau\epsilon_1\mu_1 e^{ik_\beta \cdot \vec{x}} \phi_l - e^{ik_\beta \cdot \vec{x}} \Delta\phi_l - 2ie^{ik_\beta \cdot \vec{x}} k \cdot \nabla\phi_l)\zeta.
\end{aligned}$$

Now we can estimate the term $\|(\mathcal{A}_{\tilde{\epsilon}}(\beta) - \tau)\vec{u}_l\|_{\tilde{\epsilon}}$. Since k and ϕ_l are real-valued, using the identity above we have

$$\begin{aligned}
&\|\mathcal{A}_{\tilde{\epsilon}}(\beta)\vec{u}_l - \tau\vec{u}_l\|_{\tilde{\epsilon}}^2 \\
&\quad = \mu^{-2}\|\nabla_\beta \times \nabla_\beta \times \vec{u}_l - \epsilon_1\mu_1\tau\vec{u}_l\|^2 \\
&\quad = \mu^{-2}\|(-e^{ik_\beta \cdot \vec{x}} \Delta\phi_l - 2ie^{ik_\beta \cdot \vec{x}} k \cdot \nabla\phi_l)\zeta\|^2 \\
&\quad = \mu^{-2}\|\Delta\phi_l + 2ik \cdot \nabla\phi_l\|^2 \\
&\quad = \mu^{-2}(\|\Delta\phi_l\|^2 + 4\|k \cdot \nabla\phi_l\|^2) \\
&\quad = \mu^{-2} l^{-4}\|\Delta\phi\|^2 + 4\mu^{-2} l^{-2}\|k \cdot \nabla\phi\|^2.
\end{aligned}$$

Hence, to ensure the inequality (23), the following inequality is sufficient:

$$\mu^{-2} l^{-4} \|\Delta\phi\|^2 + 4\mu^{-2} l^{-2} \|k\cdot\nabla\phi\|^2 < d^2\epsilon_1^2. \tag{29}$$

Inequality (29) is equivalent to

$$l^4 d^2 \epsilon_1^2 \mu_1^2 > \|\Delta\phi\|^2 + 4l^2 \|k\cdot\nabla\phi\|^2. \tag{30}$$

We define

$$\vec{n}_k = |k|^{-1} k = (\tau\epsilon_1\mu_1 - \beta^2)^{-\frac{1}{2}} k \tag{31}$$

as the unit vector according to k. Then we can rewrite (30) as

$$(l^2\epsilon_1\mu_1)^2 > d^{-2}\big(\|\Delta\phi\|^2 + 4(\tau l^2\epsilon_1\mu_1 - l^2\beta^2)\|\vec{n}_k\cdot\nabla\phi\|^2\big). \tag{32}$$

We will consider (32) in two different cases.

i) For $\epsilon_1\mu_1$ fixed, if

$$l^2 \geq \frac{2(\tau\epsilon_1\mu_1 - \beta^2)}{d^2\epsilon_1^2\mu_1^2} \inf_{\substack{\phi\in C_0^2(\Omega)\\ \|\phi\|=1}} \left(\|\vec{n}_k\cdot\nabla\phi\|^2 \left(1 + \sqrt{1 + \frac{d^2\epsilon_1^2\mu_1^2\|\Delta\phi\|^2}{4(\tau\epsilon_1\mu_1 - \beta^2)^2\|\vec{n}_k\cdot\nabla\phi\|^4}} \right) \right).$$

the inequality (23) holds.

ii) For l fixed, if

$$\epsilon_1\mu_1 \geq \frac{2\tau}{l^2 d^2} \inf_{\substack{\phi\in C_0^2(\Omega)\\ \|\phi\|=1}} \Bigg(\|\vec{n}_k\cdot\nabla\phi\|^2 \times \left(1 + \sqrt{\max\left\{0, 1 + \frac{d^2(\|\Delta\phi\|^2 - 4l^2\beta^2\|\vec{n}_k\cdot\nabla\phi\|^2)}{4\tau^2\|\vec{n}_k\cdot\nabla\phi\|^4}\right\}} \right) \Bigg)$$

the inequality (23) holds. Thus the theorem is proved. □

Remark 3.1 There is a very important question which is still not clear: Can eigenvalues created by line defects be embedded in the essential spectrum of the background medium? A similar argument concerning this issue in optical waveguides, the traditional counterpart of photonic crystal fibers, has been studied in [4] and [18]. A traditional optical waveguide is a dielectric medium whose cross section differs only by a compactly supported perturbation from a homogeneous reference medium. In this case, they conclude that the eigenvalues created by the perturbation can not embed in the essential spectrum of the reference medium, except possibly the lower edge of the essential spectrum. For the Schrödinger operators, similar problem has been intensively studied. See, e.g., [24] and references therein.

4 Combes-Thomas Estimates

Classical wave operators, e.g., acoustic operators and Maxwell operators, can be regarded as generalized Schrödinger operators. Usually they satisfy exponential decay estimates which are called Combes-Thomas estimates in mathematical physics. See, e.g. [1, 5, 9, 10, 14, 19, 27].

We write $\mathcal{R}(z) = (\mathcal{A}_{\epsilon_0}(\beta) - zI)^{-1}$ for $z \in \rho(\mathcal{A}_{\epsilon_0}(\beta))$ and $R(z) = (A_{\epsilon_0}(\beta) - zI)^{-1}$ for $z \in \rho(A_{\epsilon_0}(\beta))$. In the sequel, we formulate the modified Combes-Thomas estimates on the decay of the resolvent $\mathcal{R}(z)$ and the operator $\nabla_\beta \times \mathcal{R}(z)$ (see Theorems 4.1 and 4.3 below). Theorem 4.1 has been formulated in [25], the main idea of its proof is based on Lemma 12 in [9] and Lemma 15 in [10]. Then we argue as proof of Lemma 16 in [10] to prove Theorem 4.3. As we will see in Sect. 5 that these estimates are very important for studying the behavior of guided waves away from line defects. *It is worth noting that the periodic conditions* (3) *for $\epsilon_0(x)$ and $\mu_0(x)$ are unnecessary for deriving these estimates in this section.*

Let $\chi_{x,h}$ be the characteristic function of a square of side $2h$ centered at x, i.e.,

$$\chi_{x,h} = \chi_{\Omega_{x,h}}$$

where

$$\Omega_{x,h} = \{y \in \mathbb{R}^2 \mid |y_1 - x_1| \le h,\ |y_2 - x_2| \le h\}.$$

For a vector $\vartheta \in \mathbb{C}^n$, $n \in \mathbb{N}$, we set $|\vartheta| = \sqrt{\sum_{j=1}^n |\vartheta_j|^2}$. For a matrix $C = (c_{jk})$, $1 \le j \le m$, $1 \le k \le n$, we set $|C|_\infty = \max_{1\le j\le m} \sum_{k=1}^n |c_{jk}|$. And for a measurable function $f(x)$, we set $\|f\|_\infty = \operatorname{esssup}|f|$. Finally we denote by $\langle\cdot,\cdot\rangle$ the inner product of the Hilbert space H with the norm $\|\cdot\|$.

First, we give the estimate on the resolvent $R(z)$. We refer to Theorem 5.1 in [25] for the proof.

Theorem 4.1 *Let $z \in \rho(A_{\epsilon_0}(\beta))$. Then for any $n \in \mathbb{N}$, $h > 0$ and $0 < \nu < 1$, we have*

$$\|\chi_{x,h} R^n(z) \chi_{y,h}\|_{\epsilon_0} \le \left(\left(\frac{1+\nu}{1-\nu}\right)^2 \frac{1}{d}\right)^n e^{2\sqrt{2}h\nu\theta_0} e^{-\nu\theta_0|x-y|} \quad \text{for all } x, y \in \mathbb{R}^2 \tag{33}$$

with

$$\theta_0 = \frac{d}{4}\sqrt{\frac{\mu_{0,-}}{d+|z|}}, \tag{34}$$

where

$$d \equiv \operatorname{dist}(z, \sigma(A_{\epsilon_0}(\beta))) = \inf_{\vec{u}\in \operatorname{Dom}(A_{\epsilon_0}(\beta)),\ \|\vec{u}\|_{\epsilon_0}=1} \|(A_{\epsilon_0}(\beta) - zI)\vec{u}\|_{\epsilon_0}$$

and $\mu_{0,-}$ is defined in (2). *The norm in the left hand side of* (33) *is the operator norm in H_{ϵ_0}, where the definition of H_{ϵ_0} is analogous to H_ϵ in* (12).

Remark 4.1 It is worth noting that the resolvent decay exponentially fast, and the rate of exponential decay grows if d (the distance from z to the edge of $\sigma(A_{\epsilon_0}(\beta))$) grows. For Schrödinger operators, results of [1] show that the rate of exponential decay, which also depends on the distance from z to the edge of the spectrum, behaves as $\sqrt{(z-\alpha)(\beta-z)}$.

In the sequel, we formulate the estimate on the operator $\nabla_\beta \times R(z)$. However, we shall first derive some interior regularity estimates needed.

We first introduce the Hermite matrix

$$\partial_\beta := \begin{pmatrix} 0 & i\beta & \partial_2 \\ -i\beta & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \quad \text{for } \beta > 0,$$

and then formally define a vector-valued operator

$$\partial_\beta \vec{u} := \begin{pmatrix} \partial_2 u_3 + i\beta u_2 \\ -\partial_1 u_3 - i\beta u_1 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}. \tag{35}$$

The domain of ∂_β, which is denoted by $\mathrm{Dom}(\partial_\beta)$, is the closure of $C_0^\infty(\mathbb{R}^2, \mathbb{C}^3)$ in the norm

$$(\|\vec{u}\|_{\epsilon_0}^2 + \|\partial_\beta \vec{u}\|_{\epsilon_0}^2)^{\frac{1}{2}} \quad \text{for any } \vec{u} \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^3),$$

and ∂_β is a closed densely defined operator on H_{ϵ_0}. We also introduce the weighted Hilbert space

$$H_{\Omega,\epsilon_0} := \{\vec{u} \mid \vec{u} \in L^2(\Omega, \epsilon_0 dx; \mathbb{C}^3)\},$$

where Ω is an open subset of $\mathbb{R}^2$. Then we can also define the restriction of the operator ∂_β as the closed densely defined operator on the space H_{Ω,ϵ_0} by $\partial_{\beta,\Omega}$ in the same way for $\vec{u} \in C_0^\infty(\Omega, \mathbb{C}^3)$ with $\partial_{\beta,\Omega}\vec{u} \in H_{\Omega,\epsilon_0}$. For $\Omega_1 \subset \Omega$, if $\vec{u} \in \mathrm{Dom}(\partial_{\beta,\Omega})$, then $\vec{u}\mid_{\Omega_1} \in \mathrm{Dom}(\partial_{\beta,\Omega_1})$ and $\partial_{\beta,\Omega}\vec{u}\mid_{\Omega_1} = \partial_{\beta,\Omega_1}\vec{u}\mid_{\Omega_1}$. Hence we can write $\partial_{\beta,\Omega}\vec{u}$ as $\partial_\beta \vec{u}$ for simplicity.

We can easily find from (35) and (9) that $\nabla_\beta \times \vec{u} = \partial_\beta \vec{u}$ and $A_{\epsilon_0}(\beta) = \epsilon_0^{-1}\partial_\beta \mu_0^{-1}\partial_\beta$. *Thus we use $\nabla_\beta \times$ and ∂_β without difference in the following*.

For the scalar valued function $\phi \in C^\infty(\mathbb{R}^2; \mathbb{R})$, we define the matrix-valued operator $\partial_\beta *$ by

$$\partial_\beta * \phi \equiv \begin{pmatrix} 0 & i\beta\phi & \partial_2\phi \\ -i\beta\phi & 0 & -\partial_1\phi \\ -\partial_2\phi & \partial_1\phi & 0 \end{pmatrix}. \tag{36}$$

A straightforward calculation shows that

$$\nabla_\beta \times (\phi\vec{u}) = (\partial_\beta * \phi)\vec{u} + \phi\partial_\beta \vec{u}$$

for any $\phi \in C^\infty(\mathbb{R}^2; \mathbb{R})$, and $\vec{u} = (u_1, u_2, u_3) \in V_{\epsilon_0}$, where V_{ϵ_0} is defined in (13). Finally, we set

$$V_{\Omega,\epsilon_0} := \{\vec{u} \in H_{\Omega,\epsilon_0} \mid \partial_\beta \vec{u} \in H_{\Omega,\epsilon_0}\}.$$

Definition 4.1 A function $\vec{u} \in V_{\Omega,\epsilon_0}$ is said to be a *weak solution* for the equation

$$\epsilon_0^{-1}\partial_\beta \mu_0^{-1}\partial_\beta \vec{u} = \vec{w} \quad \text{in } \Omega \tag{37}$$

with $\vec{w} \in H_{\Omega,\epsilon_0}$, if it satisfy the equation

$$\langle \partial_\beta \vec{v}, \mu_0^{-1}\partial_\beta \vec{u}\rangle_\Omega = \langle \vec{v}, \vec{w}\rangle_{\Omega,\epsilon_0}, \quad \text{for all } \vec{v} \in C_0^\infty(\Omega; \mathbb{C}^3). \tag{38}$$

Then we have the following theorem:

Theorem 4.2 *Suppose $\vec{u} \in V_{\Omega,\epsilon_0}$ is a weak solution of (37) in an open subset $\Omega \subset \mathbb{R}^2$ with $\vec{w} \in H_{\Omega,\epsilon_0}$, then for any measurable subset $\Omega_0 \subset\subset \Omega$ with $\mathrm{dist}(\Omega_0, \partial\Omega) \geq \delta$ for some $\delta > 0$, there holds*

$$\langle \partial_\beta \vec{u}, \mu_0^{-1}\partial_\beta \vec{u}\rangle_{\Omega_0} \leq \kappa\left(\frac{1}{\mu_{0,-}}\|\vec{u}\|_{\Omega,\epsilon_0}^2 + \mu_{0,-}\|\vec{w}\|_{\Omega,\epsilon_0}^2\right) \tag{39}$$

where $\kappa = \kappa(\delta, \beta, \epsilon_{0,-})$.

Proof Let Ω_1 be an open set, such that

$$\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$$

with

$$\mathrm{dist}(\Omega_0, \partial\Omega_1) \geq \frac{1}{2}\delta \quad \text{and} \quad \mathrm{dist}(\Omega_1, \partial\Omega) \geq \frac{1}{4}\delta.$$

Furthermore, we can find a scalar function $\phi(x)$, such that $\phi(x) \in C_0^\infty(\Omega_1)$, $0 \leq \phi(x) \leq 1$, for $x \in \mathbb{R}^2$, $\phi(x) \equiv 1$ in Ω_0, and $\|\partial_1\phi\|_\infty < \frac{2}{\delta}$, $\|\partial_2\phi\|_\infty < \frac{2}{\delta}$. From (36) we can see

$$|\partial_\beta * \phi|_\infty \leq C_{\delta,\beta},$$

where $C_{\delta,\beta} = \beta + \frac{4}{\delta}$. Then the following part of the proof is just a modified form of [10]. Since $\phi^2\vec{u} \in V_{\Omega,\epsilon_0}$ with compact support, it follows from (38) that

$$\langle \partial_\beta(\phi^2\vec{u}), \mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega = \langle \phi^2\vec{u}, \vec{w}\rangle_{\Omega,\epsilon_0}.$$

Since

$$\langle \partial_\beta(\phi^2\vec{u}), \mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega = \langle \partial_\beta\vec{u}, \phi^2\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega + 2\langle(\partial_\beta * \phi)\vec{u}, \phi\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega.$$

Using the inequality $ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2$ for any $a, b,$ and $\gamma > 0$, then we have

$$\begin{aligned}
0 &\leq \langle \partial_\beta\vec{u}, \phi^2\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega \\
&= \langle \phi^2\vec{u}, \vec{w}\rangle_{\Omega,\epsilon_0} - 2\langle(\partial_\beta * \phi)\vec{u}, \phi\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega \\
&\leq \|\vec{u}\|_{\Omega,\epsilon_0}\|\vec{w}\|_{\Omega,\epsilon_0} + 2\langle(\partial_\beta * \phi)\vec{u}, \mu_0^{-1}(\partial_\beta * \phi)\vec{u}\rangle_\Omega^{\frac{1}{2}}\langle \partial_\beta\vec{u}, \phi^2\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega^{\frac{1}{2}} \\
&\leq \frac{1}{2\mu_{0,-}}\|\vec{u}\|_{\Omega,\epsilon_0}^2 + \frac{\mu_{0,-}}{2}\|\vec{w}\|_{\Omega,\epsilon_0}^2 + 2\frac{1}{\mu_{0,-}}C_{\delta,\beta}\frac{1}{\epsilon_{0,-}}\|\vec{u}\|_{\Omega,\epsilon_0} + \frac{1}{2}\langle \partial_\beta\vec{u}, \phi^2\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega.
\end{aligned}$$

This implies

$$\langle \partial_\beta\vec{u}, \phi^2\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega \leq \frac{1}{\mu_{0,-}}\left(1 + 4\frac{1}{\epsilon_{0,-}C_{\delta,\beta}}\right)\|u\|_{\Omega,\epsilon_0}^2 + \mu_{0,-}\|\vec{w}\|_{\Omega,\epsilon_0}^2.$$

We set $\kappa = (1 + 4\frac{1}{\epsilon_{0,-}C_{\delta,\beta}})$, then the inequality (39) follows from the fact:

$$\langle \partial_\beta\vec{u}, \mu_0^{-1}\partial_\beta\vec{u}\rangle_{\Omega_0} \leq \langle \partial_\beta\vec{u}, \phi^2\mu_0^{-1}\partial_\beta\vec{u}\rangle_\Omega.$$

□

Corollary 4.1 *For any $\vec{u} \in V_{\epsilon_0}$ be a weak solution of the equation*

$$\epsilon_0^{-1}\partial_\beta\mu_0^{-1}\partial_\beta\vec{u} = \vec{w} \quad \text{in } \mathbb{R}^2$$

where $\vec{w} \in H_{\epsilon_0}$, we have

$$\|\partial_\beta\vec{u}\| \leq \sqrt{\kappa\mu_{0,+}}\left(\frac{1}{\sqrt{\mu_{0,-}}}\|\vec{u}\|_{\epsilon_0} + \sqrt{\mu_{0,-}}\|\vec{w}\|_{\epsilon_0}\right)$$

where $\kappa = \inf_{\delta>0}\kappa_\delta$.

Now we can derive the estimate on the decay of the operator $\partial_\beta R(z)$ (i.e., the operator $\nabla_\beta \times R(z)$).

Theorem 4.3 *For any $z \in \rho(A_{\epsilon_0}(\beta))$, the operator $\partial_\beta R(z) : V_{\epsilon_0} \to V_{\epsilon_0}$ has the bound*:

$$\|\partial_\beta R(z)\| \le \frac{1}{d}\sqrt{\kappa}\left(\sqrt{\frac{\mu_{0,+}}{\mu_{0,-}}} + \sqrt{\mu_{0,-}\mu_{0,+}}(|z| + d)\right)$$

where $d = \mathrm{dist}(z, \sigma(A_{\epsilon_0}(\beta)))$. Furthermore, for any $h > 0$ and $0 < \nu < 1$, we have

$$\|\chi_{x,h}\partial_\beta R(z)\chi_{y,h}\| \le \Xi\left(\frac{1+\nu}{1-\nu}\right)^2 \frac{1}{d} e^{6\sqrt{2}h\nu\theta_0} e^{-\nu\theta_0|x-y|} \tag{40}$$

for all $x, y \in \mathbb{R}^2$ with $|x - y| \ge 2h$, where

$$\Xi = \sqrt{\kappa\mu_{0,+}}\left(\frac{1}{\sqrt{\mu_{0,-}}} + |z|\sqrt{\mu_{0,-}}\right),$$

and the definitions of θ_0 and d are the same as in Theorem 4.1. The norm in the left hand side of (40) is the operator norm in $L^2(\mathbb{R}^2; \mathbb{C}^3)$.

Proof For any $\vec{u} \in H_{\epsilon_0}$, we have $R(z)\vec{u} \in V_{\epsilon_0}$. Note that

$$A_{\epsilon_0}(\beta)R(z) = \big(I + zR(z)\big),$$

i.e.,

$$\epsilon_0^{-1}\partial_\beta\mu_0^{-1}\partial_\beta R(z) = \big(I + zR(z)\big),$$

by applying Corollary 4.1 and the inequality $||R(z)||_{\epsilon_0} \le \frac{1}{d}$, we have

$$\begin{aligned}
\|\partial_\beta R(z)\| &\le \sqrt{\kappa\mu_{0,+}}\left(\frac{1}{\sqrt{\mu_{0,-}}}\|R(z)\|_{\epsilon_0} + \sqrt{\mu_{0,-}}\|I + zR(z)\|_{\epsilon_0}\right) \\
&\le \sqrt{\kappa\mu_{0,+}}\left(\frac{1}{\sqrt{\mu_{0,-}}}\frac{1}{d} + \sqrt{\mu_{0,-}}\left(\frac{|z|}{d} + 1\right)\right) \\
&= \frac{1}{d}\sqrt{\kappa}\left(\sqrt{\frac{\mu_{0,+}}{\mu_{0,-}}} + \sqrt{\mu_{0,-}\mu_{0,+}(|z| + d)}\right).
\end{aligned}$$

Next, we shall prove (40). Let $x, y \in \mathbb{R}^2$ and $h > 0$ with $|x - y| > 2h$, one can see that $\kappa_{x,3h}\kappa_{y,h} \equiv 0$. Thus for any $\vec{u} \in H_{\epsilon_0}$, by applying Corollary 4.1, we have

$$\begin{aligned}
&\|\chi_{x,h}\partial_\beta R(z)\chi_{y,h}\vec{u}\| \\
&\le \sqrt{\kappa\mu_{0,+}}\left(\frac{1}{\sqrt{\mu_{0,-}}}\|\chi_{x,3h}R(z)\chi_{y,h}\vec{u}\|_{\epsilon_0} + \sqrt{\mu_{0,-}}\|\chi_{x,3h}(I + zR(x))\chi_{y,h}\vec{u}\|_{\epsilon_0}\right) \\
&\le \sqrt{\kappa\mu_{0,+}}\left(\left(\frac{1}{\sqrt{\mu_{0,-}}} + |z|\sqrt{\mu_{0,-}}\right)\|\chi_{x,3h}R(z)\chi_{y,h}\vec{u}\|_{\epsilon_0} + \sqrt{\mu_{0,-}}\|\chi_{x,3h}\chi_{y,h}\vec{u}\|_{\epsilon_0}\right) \\
&= \sqrt{\kappa\mu_{0,+}}\left(\frac{1}{\sqrt{\mu_{0,-}}} + |z|\sqrt{\mu_{0,-}}\right)\|\chi_{x,3h}R(z)\chi_{y,h}\vec{u}\|_{\epsilon_0}.
\end{aligned}$$

We can further apply Theorem 4.1 to obtain

$$\begin{aligned}\|\chi_{x,h}\partial_\beta R(z)\chi_{y,h}\| &\le \Xi\|\chi_{x,3h}R(z)\chi_{y,h}\| \\ &\le \Xi\|\chi_{x,3h}R(z)\chi_{y,3h}\| \\ &\le \Xi\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}e^{6\sqrt{2}h\nu\theta_0}e^{-\nu\theta_0|x-y|}\end{aligned}$$

for all $x, y \in \mathbb{R}^2$ and $|x-y| \ge 2h$, where $\Xi = \sqrt{\kappa\mu_{0,+}}(\frac{1}{\sqrt{\mu_{0,-}}} + |z|\sqrt{\mu_{0,-}})$. □

Note that $\sigma(\mathcal{A}_{\epsilon_0}(\beta)) = \sigma(A_{\epsilon_0}(\beta))\bigcup\{0\}$, we are fortunate to see that not only the conclusions but also the proofs of the Combes-Thomas estimates on $R(z)$ hold for their counterparts on $\mathcal{R}(z) = (\mathcal{A}_{\epsilon_0}(\beta) - zI)^{-1}$ by carefully checking the proofs of Theorems 4.1 and 4.3. More precisely, we have the following two theorems:

Theorem 4.4 *Let $z \in \rho(\mathcal{A}_{\epsilon_0}(\beta))$. Then for any $n \in \mathbb{N}$, $h > 0$ and $0 < \nu < 1$, we have*

$$\|\chi_{x,h}\mathcal{R}^n(z)\chi_{y,h}\|_{\epsilon_0} \le \left(\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}\right)^n e^{2\sqrt{2}h\nu\theta_0}e^{-\nu\theta_0|x-y|} \quad \text{for all } x, y \in \mathbb{R}^2 \tag{41}$$

with

$$\theta_0 = \frac{d}{4}\sqrt{\frac{\mu_{0,-}}{d+|z|}},$$

where

$$d = \operatorname{dist}(z, \sigma(\mathcal{A}_{\epsilon_0}(\beta))) = \inf_{\vec{u}\in \operatorname{Dom}(\mathcal{A}_{\epsilon_0}(\beta)),\ \|\vec{u}\|_{\epsilon_0}=1}\|(\mathcal{A}_{\epsilon_0}(\beta) - zI)\vec{u}\|_{\epsilon_0}$$

and $\mu_{0,-}$ is defined in (2). The norm in the left hand side of (41) is the operator norm in H_{ϵ_0}, where H_{ϵ_0} is defined in (12).

Theorem 4.5 *For any $z \in \rho(\mathcal{A}_{\epsilon_0}(\beta))$, the operator $\partial_\beta\mathcal{R}(z) : V_{\epsilon_0} \to V_{\epsilon_0}$ has the bound:*

$$\|\partial_\beta\mathcal{R}(z)\| \le \frac{1}{d}\sqrt{\kappa}\left(\sqrt{\frac{\mu_{0,+}}{\mu_{0,-}}} + \sqrt{\mu_{0,-}\mu_{0,+}}(|z|+d)\right)$$

where $d = \operatorname{dist}(z, \sigma(\mathcal{A}_{\epsilon_0}(\beta)))$ and $\mu_{0,\pm}$ is defined in (2). Furthermore, for any $h > 0$ and $0 < \nu < 1$, we have

$$\|\chi_{x,h}\partial_\beta\mathcal{R}(z)\chi_{y,h}\| \le \Xi\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}e^{6\sqrt{2}h\nu\theta_0}e^{-\nu\theta_0|x-y|} \tag{42}$$

for all $x, y \in \mathbb{R}^2$ with $|x-y| \ge 2h$, where

$$\Xi = \sqrt{\kappa\mu_{0,+}}\left(\frac{1}{\sqrt{\mu_{0,-}}} + |z|\sqrt{\mu_{0,-}}\right),$$

and the definitions of θ_0 and d are the same as those in Theorem 4.4. The norm in the left hand side of (42) is the operator norm in $L^2(\mathbb{R}^2; \mathbb{C}^3)$.

5 The Exponential Decay of Guided Waves Away from Line Defects

In this section we show that any eigenfunctions created by a line defect strip decay exponentially away from the defect strip. Following Sect. 3, we describe the background medium by ϵ_0 and μ_0, and the perturbed medium by $\tilde{\epsilon}(x)$ and $\tilde{\mu}(x)$, we also adapt $\mathcal{A}_{\tilde{\epsilon}}(\beta)$ as the perturbed operator according to $\mathcal{A}_{\epsilon_0}(\beta)$.

Let B be a spectral gap of $\mathcal{A}_{\epsilon_0}(\beta)$ and $z \in \sigma(\mathcal{A}_{\tilde{\epsilon}}(\beta)) \cap B$. It follows from Theorem 2.2 that z must be an eigenvalue with finite multiplicity of the operator $\mathcal{A}_{\tilde{\epsilon}}(\beta)$.

Theorem 5.1 *Let $z \in \sigma(\mathcal{A}_{\tilde{\epsilon}}(\beta)) \cap B$ and $\vec{u}$ be the corresponding eigenfunction, then we have*

$$\|\chi_{x,h}\vec{u}\|_{\tilde{\epsilon}} \le C_0 e^{-\frac{1}{2}\nu\theta_0|x-y|}$$

for all $x, y \in \mathbb{R}^2$ with $|x-y| \ge 2h$, and $0 < \nu < 1$, where $\chi_{x,h}$ is the characteristic function of the set $\{y||x-y| \le h\}$ with $h > 0$, the constant C_0 depends on $\nu, z, \epsilon_0, \mu_0, \epsilon, \mu$ and the distance from z to the edge of $\sigma(\mathcal{A}_{\epsilon_0}(\beta))$.

To prove this theorem, the following lemma is needed.

Lemma 5.1 *Let $z \in \sigma(\mathcal{A}_{\tilde{\epsilon}}(\beta)) \cap B$ and $\vec{u}$ be the corresponding eigenfunction of the operator $\mathcal{A}_{\tilde{\epsilon}}(\beta)$, then we have*

$$\|\nabla_\beta \times \vec{u}\| \le \sqrt{z\tilde{\epsilon}_+^2\tilde{\mu}_+}\|\vec{u}\|_{\tilde{\epsilon}},$$

where $\tilde{\epsilon}_+$ and $\tilde{\mu}_+$ are defined by

$$\tilde{\epsilon}_+ = \sup_{x\in\mathbb{R}^2} \tilde{\epsilon}(x), \qquad \tilde{\mu}_+ = \sup_{x\in\mathbb{R}^2} \tilde{\mu}(x).$$

Proof Since $\mathcal{A}_{\tilde{\epsilon}}(\beta) + I$ is strictly positive on the weighted space $H_{\tilde{\epsilon}}$, we can rewrite the equation

$$\mathcal{A}_{\tilde{\epsilon}}(\beta)\vec{u} = z\vec{u}, \qquad \vec{u} \in \mathrm{Dom}\big(\mathcal{A}_{\tilde{\epsilon}}(\beta)\big)$$

as

$$\vec{u} = (z+1)(\mathcal{A}_{\tilde{\epsilon}}^{-1}(\beta) + I)^{-1}\vec{u}.$$

Then we have

$$\begin{aligned}
\|\nabla_\beta \times \vec{u}\|_{\tilde{\epsilon}}^2 &= (z+1)\langle\nabla_\beta \times \vec{u}, \nabla_\beta \times (\mathcal{A}_{\tilde{\epsilon}}^{-1}(\beta) + I)^{-1}\vec{u}\rangle_{\tilde{\epsilon}} \\
&\le (z+1)\tilde{\mu}_+\langle\nabla_\beta \times \vec{u}, \tilde{\mu}^{-1}\nabla_\beta \times (\mathcal{A}_{\tilde{\epsilon}}^{-1}(\beta) + I)^{-1}\vec{u}\rangle_{\tilde{\epsilon}} \\
&= (z+1)\tilde{\mu}_+\langle\vec{u}, \nabla_\beta \times \tilde{\mu}^{-1}\nabla_\beta \times (\mathcal{A}_{\tilde{\epsilon}}^{-1}(\beta) + I)^{-1}\vec{u}\rangle_{\tilde{\epsilon}} \\
&= (z+1)\tilde{\mu}_+\langle\vec{u}, \tilde{\epsilon}\mathcal{A}_{\tilde{\epsilon}}(\beta)(\mathcal{A}_{\tilde{\epsilon}}^{-1}(\beta) + I)^{-1}\vec{u}\rangle_{\tilde{\epsilon}} \\
&\le (z+1)\tilde{\epsilon}_+\tilde{\mu}_+\langle\vec{u}, \vec{u}\rangle_{\tilde{\epsilon}}.
\end{aligned}$$

This implies that

$$\|\nabla_\beta \times \vec{u}\|_{\tilde{\epsilon}} \le \sqrt{(z+1)\tilde{\epsilon}_+\tilde{\mu}_+}\|\vec{u}\|_{\tilde{\epsilon}},$$

i.e.,

$$\|\nabla_\beta \times \vec{u}\| \le \sqrt{(z+1)\tilde{\epsilon}_+^2 \tilde{\mu}_+}\|\vec{u}\|_{\tilde{\epsilon}}.$$

□

Remark 5.1 We can also use Corollary 4.1 to obtain a similar estimate.

Then we shall complete the proof of Theorem 5.1.

Proof As in [11, 23], we first introduce a quadratic form:

$$a[\vec{v}, \vec{u}] := \langle \mathcal{A}_{\epsilon_0}(\beta)\vec{v}, \vec{u}\rangle_{\epsilon_0} - z\langle \vec{v}, \vec{u}\rangle_{\epsilon_0}$$

for all $\vec{v} \in \mathrm{Dom}(\mathcal{A}_{\epsilon_0}(\beta))$, and $\vec{u} \in \mathrm{Dom}(\mathcal{A}_{\tilde{\epsilon}}(\beta))$. Let $\vec{v} = \mathcal{R}(z)\chi_x\vec{u}$, where $\mathcal{R}(z) = (\mathcal{A}_{\epsilon_0}(\beta) - zI)^{-1}$, then we have

$$\begin{aligned} a[\vec{v}, \vec{u}] &= \langle (\mathcal{A}_{\epsilon_0}(\beta) - zI)\vec{v}, \vec{u}\rangle_{\epsilon_0} \\ &= \langle \chi_x\vec{u}, \vec{u}\rangle_{\epsilon_0} \\ &= \|\chi_x\vec{u}\|_{\epsilon_0}^2 \\ &\le \epsilon_{0,+}\|\chi_x\vec{u}\|^2. \end{aligned} \tag{43}$$

On the other hand, using $\mathcal{A}_{\tilde{\epsilon}}(\beta)\vec{u} = z\vec{u}$, we get

$$\begin{aligned} a[\vec{v}, \vec{u}] &= \langle \epsilon_0^{-1}\nabla_\beta \times \mu_0^{-1}\nabla_\beta \times \vec{v}, \vec{u}\rangle_{\epsilon_0} - z\langle \vec{v}, \vec{u}\rangle_{\epsilon_0} \\ &= \langle \nabla_\beta \times \vec{v}, \mu_0^{-1}\nabla_\beta \times \vec{u}\rangle - z\langle \vec{v}, \vec{u}\rangle_{\epsilon_0} \\ &= \langle \nabla_\beta \times \vec{v}, \mu_0^{-1}\nabla_\beta \times \vec{u}\rangle - z\langle \vec{v}, \vec{u}\rangle_{\epsilon_0} \\ &\quad + \left\langle \vec{v}, \frac{\epsilon}{\epsilon_0}\mathcal{A}_\epsilon(\beta)\vec{u}\right\rangle_{\epsilon_0} - \left\langle \vec{v}, \frac{\epsilon}{\epsilon_0}\mathcal{A}_\epsilon(\beta)\vec{u}\right\rangle_{\epsilon_0} \\ &= -\langle \nabla_\beta \times \vec{v}, \eta\nabla_\beta \times \vec{u}\rangle + \langle \nabla_\beta \times \vec{v}, \mu^{-1}\nabla_\beta \times \vec{u}\rangle \\ &\quad - z\langle \vec{v}, \vec{u}\rangle_{\epsilon_0} + \langle \vec{v}, \epsilon\xi\mathcal{A}_\epsilon(\beta)\vec{u}\rangle_{\epsilon_0} + \langle \vec{v}, z\vec{u}\rangle_{\epsilon_0} - \left\langle \vec{v}, \frac{\epsilon}{\epsilon_0}\mathcal{A}_\epsilon(\beta)\vec{u}\right\rangle_{\epsilon_0} \\ &= -\langle \nabla_\beta \times \vec{v}, \eta\nabla_\beta \times \vec{u}\rangle - \langle \vec{v}, \epsilon\xi\mathcal{A}_\epsilon(\beta)\vec{u}\rangle_{\epsilon_0} + \langle \nabla_\beta \times \vec{v}, \mu^{-1}\nabla_\beta \times \vec{u}\rangle \\ &\quad - \langle \vec{v}, \nabla_\beta \times \mu^{-1}\nabla_\beta \times \vec{u}\rangle \\ &= -\langle \nabla_\beta \times \vec{v}, \eta\nabla_\beta \times \vec{u}\rangle - \langle \vec{v}, z\epsilon\xi\vec{u}\rangle_{\epsilon_0}. \end{aligned}$$

Taking account of Lemma 5.1, Theorems 4.4 and 4.5, we have

$$\begin{aligned} |a[\vec{v}, \vec{u}]| &\le \|\chi_\eta\nabla_\beta \times \vec{v}\|\|\eta\|_\infty\|\nabla_\beta \times \vec{u}\| + z\|\epsilon - \epsilon_0\|_\infty\|\chi_\xi\vec{v}\|\|\vec{u}\| \\ &\le C_1(\|\chi_\eta\nabla_\beta \times \mathcal{R}(z)\chi_x\| + \|\chi_\xi\mathcal{R}(z)\chi_x\|) \\ &\le C_2 e^{-\nu\theta_0|x-y|}. \end{aligned} \tag{44}$$

Thus Theorem 5.1 follows from (43) and (44). □

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